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# Geometry of the Quantum Complex Projective Space $CP_q(N)$ \*

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## Abstract

The quantum deformation  $CP_q(N)$  of complex projective space is discussed. Many of the features present in the case of the quantum sphere can be extended. The differential and integral calculus is studied and  $CP_q(N)$  appears as a quantum Kähler manifold. The braiding of several copies of  $CP_q(N)$  is introduced and the anharmonic ratios of four collinear points are shown to be invariant under quantum projective transformations. They provide the building blocks of all projective invariants. The Poisson limit is also described.

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# 1 Introduction

In a recent paper [1] the quantum sphere was described as a complex quantum manifold. Then, in [2], the braiding of several copies of the quantum sphere was introduced and quantum anharmonic ratios (cross ratios) of four points on the sphere were defined which are invariant under the fractional transformation which describes the coaction of the quantum group  $SU_q(2)$  on the complex coordinates  $z, \bar{z}$  on the quantum sphere. In the present paper we extend the results of [1] and [2] to higher dimensions. In Secs. 2 and 3 we define the quantum projective space  $CP_q(N)$  in terms of both homogeneous and inhomogeneous complex coordinates and we study the differential calculus on it.  $CP_q(N)$  is shown to be the quantum deformation of a Kähler manifold with the Fubini-Study metric. In Sec. 4 we consider the Poisson limit. In Sec. 5 we study the integration of functions on  $CP_q(N)$  and give explicit formulas for the integrals. Then, in Sec. 6 we introduce the braiding of several copies of  $CP_q(N)$  and in Sec. 7 we study the anharmonic ratio of four collinear points in  $CP_q(N)$ . Just as in the classical case these anharmonic ratios can be shown to be the building blocks of all invariants under quantum projective transformations. For this reasons we have given Sec. 7 the title “Quantum Projective Geometry”.

All formulas and derivations of [1] and [2] can be easily modified, with a few changes of signs, to describe the quantum unit disk and the coaction of quantum  $SU_q(1,1)$  on it, as well as the corresponding invariant anharmonic ratios. This provides a quantum deformation of the Bolyai-Lobachevskii non-Euclidean geometry and of the differential calculus on the Bolyai-Lobachevskii plane. We shall not write here the modified equations appropriate for this case, which can be guessed very easily, but we would like to mention that the commutation relations between the variables  $z$  and  $\bar{z}$  for the unit disk are appropriate for a representation of  $z$  and  $\bar{z}$  as bounded operators in a Hilbert space. This is to be contrasted with the case of the quantum sphere where  $z$  and  $\bar{z}$  must be unbounded operators. In a perfectly

analogous way all formulas and derivations of the present paper can be easily modified, with a few changes of sign, to describe a quantum deformation of various higher dimensional non-Euclidean geometries. Again we shall not do this explicitly here and leave it as an exercise for the reader.

A different deformed algebra of functions on the Bolyai-Lobachevskii plane has been considered in [3]. The algebra of functions on complex projective space has been considered by a number of authors, see for example [4], [5] and [6]. What we have shown here is that a rich construction of differential geometry and projective geometry can be carried out on this space. It is not hard to extend most of the results of the present paper to the case of quantum Grassmannian manifolds.

## 2 $CP_q(N)$ as a Complex Manifold

### 2.1 Complex Quantum Space Covariant Under $SU_q(N+1)$

For completeness, we list here the formulas we shall need to construct the complex projective space. Remember that the  $SU_q(N+1)$  symmetry can be represented [7] on the complex quantum space  $C_q^{N+1}$  with coordinates  $x_i, \bar{x}^i, i = 0, 1, \dots, N$  which satisfy the relations

$$x_i x_j = q^{-1} \tilde{R}_{ij}^{kl} x_k x_l, \quad (1)$$

$$\bar{x}^i x_j = q(\tilde{R}^{-1})_{jl}^{ik} x_k \bar{x}^l, \quad (2)$$

$$\bar{x}^i \bar{x}^j = q^{-1} \tilde{R}_{lk}^{ji} \bar{x}^k \bar{x}^l. \quad (3)$$

Here  $q$  is a real number,  $\tilde{R}_{ij}^{kl}$  is the  $GL_q(N+1)$   $\hat{R}$ -matrix with indices running from 0 to  $N$ , and  $\bar{x}^i = x_i^*$  is the  $*$ -conjugate of  $x_i$ . The Hermitian length

$$L = x_i \bar{x}^i \quad (4)$$

is real and central.

Derivatives  $D^i, \bar{D}_i$  can be introduced (the usual symbols  $\partial^a, \bar{\partial}_b$  are reserved below for the derivatives on  $CP_q(N)$ ) which satisfy

$$D^i x_j = \delta_j^i + q \tilde{R}_{jl}^{ik} x_k D^l, \quad D^i \bar{x}^j = q (\tilde{R}^{-1})_{lk}^{ji} \bar{x}^k D^l, \quad (5)$$

$$\bar{D}_i \bar{x}^j = \delta_j^i + q^{-1} (\tilde{R}^{-1})_{ki}^{lj} \bar{x}^k \bar{D}_l, \quad \bar{D}_i x_j = q^{-1} \tilde{\Phi}_{ji}^{lk} x_k \bar{D}_l \quad (6)$$

and

$$D^i D^j = q^{-1} \tilde{R}_{lk}^{ji} D^k D^l, \quad (7)$$

$$D^i \bar{D}_j = q^{-1} \tilde{\Phi}_{lj}^{ki} \bar{D}_k D^l, \quad (8)$$

$$\bar{D}_i \bar{D}_j = q^{-1} \tilde{R}_{ij}^{kl} \bar{D}_k \bar{D}_l. \quad (9)$$

Here we have defined

$$\tilde{\Phi}_{kl}^{ij} = \tilde{R}_{lk}^{ji} q^{2(i-l)} = \tilde{R}_{lk}^{ji} q^{2(j-k)} \quad (10)$$

which satisfies

$$\tilde{\Phi}_{sj}^{ri} (\tilde{R}^{-1})_{il}^{jk} = (\tilde{R}^{-1})_{sj}^{ri} \tilde{\Phi}_{il}^{jk} = \delta_l^r \delta_s^k \quad (11)$$

and (sum over the index  $k$ )

$$\tilde{\Phi}_{jk}^{ik} = \delta_j^i q^{2i+1}, \quad (12)$$

$$\tilde{\Phi}_{kj}^{ki} = \delta_j^i q^{2(N-i)+1}. \quad (13)$$

There is a symmetry of this algebra:

$$q \rightarrow q^{-1}, \quad (14)$$

$$x_i \rightarrow q^{-2i} \bar{x}^i, \quad (15)$$

$$\bar{x}^i \rightarrow x_i, \quad (16)$$

$$D^i \rightarrow q^{2i} \bar{D}_i \quad (17)$$

and

$$\bar{D}_i \rightarrow D^i. \quad (18)$$

Exchanging the barred and unbarred quantities in (14) - (18) we get another symmetry which is the inverse of this one.

Using the fact that  $L$  commutes with  $x_i, \bar{x}^i$ , a  $*$ -involution can be defined for  $D^i$

$$(D^i)^* = -q^{-2i'} L^n \bar{D}_i L^{-n}, \quad (19)$$

where

$$i' = N - i + 1 \quad (20)$$

for any real number  $n$ . The  $*$ -involutions corresponding to different  $n$ 's are related to one another by the symmetry of conjugation by  $L$

$$a \rightarrow L^m a L^{-m}, \quad (21)$$

where  $a$  can be any function or derivative and  $m$  is the difference in the  $n$ 's.

The differentials  $\xi_i = dx_i, \bar{\xi}^i = (\xi_i)^*$  satisfy:

$$x_i \xi_j = q \tilde{R}_{ij}^{kl} \xi_k x_l, \quad (22)$$

$$\bar{x}^i \xi_j = q (\tilde{R}^{-1})_{jl}^{ik} \xi_k \bar{x}^l \quad (23)$$

and

$$\xi_i \xi_j = -q \tilde{R}_{ij}^{kl} \xi_k \xi_l, \quad (24)$$

$$\bar{\xi}^i \xi_j = -q (\tilde{R}^{-1})_{jl}^{ik} \xi_k \bar{\xi}^l. \quad (25)$$

All the above relations are covariant under the transformation

$$x_i \rightarrow x_j T_i^j, \quad \bar{x}^i \rightarrow (T^{-1})_j^i \bar{x}^j, \quad (26)$$

$$D^i \rightarrow (T^{-1})_j^i D^j, \quad \bar{D}_i \rightarrow \bar{D}_j q^{2i'} T_i^j q^{-2j'}, \quad (27)$$

$$\xi_i \rightarrow \xi_j T_i^j, \quad \bar{\xi}^i \rightarrow (T^{-1})_j^i \bar{\xi}^j, \quad (28)$$

where  $T_j^i \in SU_q(N+1)$ .

The exterior derivatives  $\delta = \xi^i D_i, \bar{\delta} = \bar{\xi}^i \bar{D}_i$  on the holomorphic and antiholomorphic functions satisfy the undeformed Leibniz rule,  $\delta^2 = \bar{\delta}^2 = 0$  and  $\bar{\delta} x_j = x_j \bar{\delta}$  etc.

## 2.2 Projective Space $CP_q(N)$

Define for  $a = 1, \dots, N$ ,<sup>‡</sup>

$$z_a = x_0^{-1} x_a, \quad \bar{z}^a = \bar{x}^a (\bar{x}^0)^{-1}. \quad (29)$$

Since

$$x_0 x_a = q x_a x_0, \quad x_0 \bar{x}^0 = \bar{x}^0 x_0, \quad (30)$$

and

$$x_0 \bar{x}^a = q^{-1} \bar{x}^a x_0, \quad (31)$$

it follows from (1) and (2) that

$$z_a z_b = q^{-1} \hat{R}_{ab}^{ce} z_c z_e, \quad (32)$$

$$\bar{z}^a z_b = q^{-1} (\hat{R}^{-1})_{be}^{ac} z_c \bar{z}^e - \lambda q^{-1} \delta_b^a. \quad (33)$$

where  $\hat{R}_{be}^{ac}$  is the  $GL_q(N)$   $\hat{R}$ -matrix with indices running from 1 to  $N$  and  $\lambda = q - 1/q$ .

Since

$$dz_a = x_0^{-1} (\xi_a - \xi_0 z_a), \quad d\bar{z}^a = (\bar{\xi}^a - \bar{z}^a \bar{\xi}^0) (\bar{x}^0)^{-1}, \quad (34)$$

and

$$x_0 \xi_0 = q^2 \xi_0 x_0, \quad x_0 \bar{\xi}^0 = \bar{\xi}^0 x_0, \quad (35)$$

it follows from (22) and (23) that

$$z_a dz_b = q \hat{R}_{ab}^{ce} dz_c z_e, \quad (36)$$

$$\bar{z}^a dz_b = q^{-1} (\hat{R}^{-1})_{be}^{ac} dz_c \bar{z}^e, \quad (37)$$

$$dz_a dz_b = -q \hat{R}_{ab}^{ce} dz_c dz_e \quad (38)$$

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<sup>‡</sup>The letters  $a, b, c, e$  etc. run from 1 to  $N$ , while  $i, j, k, l$  run from 0 to  $N$ .

and

$$d\bar{z}^a dz_b = -q^{-1}(\hat{R}^{-1})_{be}^{ac} dz_c d\bar{z}^e. \quad (39)$$

The derivatives  $\partial^a, \bar{\partial}_a$  are defined by requiring  $\delta \equiv dz_a \partial^a$  and  $\bar{\delta} \equiv d\bar{z}^a \bar{\partial}_a$  to be exterior differentiations. It follows from (36) and (37) that

$$\partial^a z_b = \delta_b^a + q \hat{R}_{be}^{ac} z_c \partial^e, \quad (40)$$

$$\partial^a \bar{z}^b = q^{-1}(\hat{R}^{-1})_{ec}^{ba} \bar{z}^c \partial^e, \quad (41)$$

$$\bar{\partial}_a z_b = q \Phi_{ba}^{ec} z_c \bar{\partial}_e, \quad (42)$$

$$\bar{\partial}_a \bar{z}^b = \delta_a^b + q^{-1}(\hat{R}^{-1})_{ca}^{eb} \bar{z}^c \bar{\partial}_e, \quad (43)$$

$$\partial^b \partial^a = q^{-1} \hat{R}_{ce}^{ab} \partial^e \partial^c \quad (44)$$

and

$$\partial^a \bar{\partial}_b = q \Phi_{eb}^{ca} \bar{\partial}_c \partial^e, \quad (45)$$

where the  $\Phi$  matrix is defined by

$$\Phi_{db}^{ca} = \hat{R}_{bd}^{ac} q^{2(c-b)} = \hat{R}_{bd}^{ac} q^{2(d-a)}. \quad (46)$$

Similarly as in the case of quantum spaces the algebra of the differential calculus on  $CP_q(N)$  has the symmetry:

$$q \rightarrow q^{-1}, \quad (47)$$

$$z_a \rightarrow q^{-2a} \bar{z}^a, \quad (48)$$

$$\bar{z}^a \rightarrow z_a, \quad (49)$$

$$\partial^a \rightarrow q^{2a} \bar{\partial}_a, \quad (50)$$

and

$$\bar{\partial}_a \rightarrow \partial^a. \quad (51)$$

Also the  $*$ -involutions

$$z_a^* = \bar{z}^a, \quad (52)$$

$$dz_a^* = d\bar{z}^a, \quad (53)$$

and

$$\partial^{a*} = -q^{2n-2a'} \rho^n \bar{\partial}_a \rho^{-n}, \quad (54)$$

where

$$a' = N - a + 1, \quad (55)$$

and

$$\rho = 1 + \sum_{a=1}^N z_a \bar{z}^a, \quad (56)$$

can be defined for any  $n$ . Corresponding to different  $n$ 's they are related with one another by the symmetry of conjugation by  $\rho$  to some powers followed by a recaling by appropriate powers of  $q$ .

In particular, the choice  $n = N + 1$  gives the  $*$ -involution which has the correct classical limit of Hermitian conjugation with the standard measure  $\rho^{-(N+1)}$  of  $CP(N)$ .

The transformation (26) induces a transformation on  $CP_q(N)$

$$z_a \rightarrow (T_0^0 + z_b T_0^b)^{-1} (T_a^0 + z_b T_a^b). \quad (57)$$

One can then calculate how the differentials transform

$$dz_a \rightarrow dz_b M_a^b, \quad d\bar{z}^a \rightarrow (M^\dagger)_b^a d\bar{z}^b \quad (58)$$

where  $M_a^b$  is a matrix of function in  $z_a$  with coefficients in  $SU_q(N+1)$  and  $(M^\dagger)_b^a \equiv (M_a^b)^*$ . Since  $\delta, \bar{\delta}$  are invariant, it follows the transformation on the derivatives

$$\partial^a \rightarrow (M^{-1})_b^a \partial^b, \quad (\partial^a)^* \rightarrow (\partial^b)^* ((M^\dagger)^{-1})_a^b \quad (59)$$

The covariance of the  $CP_q(N)$  relations under the transformation (57), (58) and (59) follows directly from the covariance in  $C_q^{N+1}$ .

### 3 A Note on the Differential Calculus

In [1], we showed that there exists a one form representation of the differential. The construction there can be generalized. Let  $A$  be a  $*$ -involutive algebra with coordinates  $z_i, \bar{z}_i$  and differentials  $dz_i, d\bar{z}_i$  such that



$\bar{z}_i = z_i^*, d\bar{z}_i = (dz_i)^*$ . If there exists a real element  $a \in A$  and real unequal nonvanishing constants  $r, s$  such that

$$az_i = rz_ia, \quad adz_i = sdz_ia, \quad \forall i, \quad (60)$$

then, as easily seen,

$$\lambda\delta f = [\eta, f]_{\pm}, \quad \eta = \frac{\lambda}{1-s/r}\delta aa^{-1}, \quad (61)$$

$$\lambda\bar{\delta}f = [\bar{\eta}, f]_{\pm} \quad \bar{\eta} = \frac{\lambda}{1-r/s}\bar{\delta}aa^{-1}, \quad (62)$$

and

$$\lambda df = [\Xi, f]_{\pm}, \quad \Xi = \eta + \bar{\eta}, \quad (63)$$

where  $\pm$  applies for odd/even forms  $f$ . Notice that (61) and (62), and therefore (60), imply that

$$ra\delta a = s\delta aa, \quad r\bar{\delta}aa = sa\bar{\delta}a. \quad (64)$$

It can be proved that  $\eta^* = -\bar{\eta}$  and so  $\Xi^* = -\Xi$ . It holds that  $\eta^2 = \bar{\eta}^2 = 0$ . However  $\Xi^2 = \eta\bar{\eta} + \bar{\eta}\eta = \lambda\delta\bar{\eta} = \lambda\bar{\delta}\eta$  will generally be nonzero. Note that

$$\lambda d\Xi = [\Xi, \Xi]_+ = 2\Xi^2. \quad (65)$$

Define

$$K = \delta\bar{\eta} = \bar{\delta}\eta \quad (66)$$

then

$$K = \frac{1}{2}d\Xi. \quad (67)$$

It follows that  $dK = 0$  and  $K^* = K$ . Thus in the case  $K \neq 0$ , we will call it a Kähler form and  $K^n$  <sup>§</sup> will be non-zero and define a real volume element

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<sup>§</sup> $n$  = complex dimension of the algebra. We consider only deformations such that the Poincaré series of the deformed algebra and its classical counterpart match.

for an integral (invariant integral if  $K^n$  is invariant).  $K$  also has the very nice property of commuting with everything

$$Kz_a = z_a K, \quad Kdz_a = dz_a K. \quad (68)$$

In the case of  $S_q^2$ ,  $K$  is just the area element.

Such a one-form representation for the calculus exists on both  $C_q^{N+1}$  and  $CP_q(N)$ . For  $C_q^{N+1}$ , we have

$$Lx_i = x_i L, \quad L\xi_i = q^2 \xi_i L \quad (69)$$

and so by taking  $a = L$ , we have

$$\eta_0 = -q^{-1} \delta L L^{-1}, \quad \bar{\eta}_0 = q \bar{\delta} L L^{-1}. \quad (70)$$

In this case,  $K$  is not the Kähler form one usually assigns to  $C_q^{N+1}$ . Rather, it gives  $C_q^{N+1}$  the geometry of  $CP_q(N)$  written in homogeneous coordinates.

Similar relations hold for  $CP_q(N)$  in inhomogeneous coordinates. It is

$$\rho z_a = q^{-2} z_a \rho, \quad \rho dz_a = dz_a \rho \quad (71)$$

and therefore

$$\eta = -q^{-1} \delta \rho \rho^{-1}, \quad \bar{\eta} = q \bar{\delta} \rho \rho^{-1}. \quad (72)$$

One can then compute

$$K = \bar{\delta} \eta \quad (73)$$

$$= dz_a g^{a\bar{b}} d\bar{z}^b, \quad (74)$$

where the metric  $g^{a\bar{b}}$  is

$$g^{a\bar{b}} = q^{-1} \rho^{-2} (\rho \delta_{ab} - q^2 \bar{z}^a z_b) \quad (75)$$

with inverse  $g_{\bar{b}c}$

$$g_{\bar{b}c} g^{c\bar{a}} = g^{a\bar{c}} g_{\bar{c}b} = \delta_{ab} \quad (76)$$

given by

$$g_{\bar{b}c} = q\rho(\delta_{bc} + \bar{z}^b z_c). \quad (77)$$

This metric is the quantum deformation of the standard Fubini-Study metric for  $CP(N)$ .

Notice that under the transformation (57)

$$\eta \rightarrow \eta + qf^{-1}\delta f, \quad f = T_0^0 + z_b T_0^b \quad (78)$$

and so  $K$  is invariant. From (58) and (74), it follows that

$$g^{a\bar{b}} \rightarrow (M^{-1})^a_c g^{c\bar{d}} ((M^\dagger)^{-1})^{\bar{d}}_{\bar{b}}, \quad (79)$$

$$g_{\bar{b}a} \rightarrow (M^\dagger)^{\bar{b}}_d g_{\bar{d}c} M^c_a. \quad (80)$$

One can show that the volume element  $dv_x$  in  $C_q^{N+1}$

$$dv_x \equiv \Pi_{j=0}^N (\bar{\xi}^j L^{-1/2}) \Pi_{i=0}^N (L^{-1/2} \xi_i) \quad (81)$$

$$= \rho^{-(N+1)} d\bar{z}^N \dots d\bar{z}^1 dz_1 \dots dz_N \cdot \bar{\xi}^0 (\bar{x}^0)^{-1} (x_0)^{-1} \xi_0. \quad (82)$$

Since  $dv_x$  is invariant, one can prove that

$$dv_z \equiv \rho^{-(N+1)} d\bar{z}^N \dots d\bar{z}^1 dz_1 \dots dz_N \quad (83)$$

is invariant also and is in fact equal to  $K^N$  (up to a numerical factor). The factor  $\rho^{-(N+1)}$  justifies the choice  $n = N + 1$  for the involution (54).

Having a quantum Kähler metric one can define connections, curvature, a Ricci tensor and a Hodge star operation. We shall not do it here because there seems to be no unique way to define these constructs. Still, once certain choices are made, the full differential geometry can be developed. See [8] for a discussion of the quantum Riemannian case.

## 4 Poisson Structures on $CP(N)$

The commutation relations in the previous sections give us, in the limit  $q \rightarrow 1$ , a Poisson structure on  $CP(N)$ . As usual, the Poisson Brackets

(P.B.s) are obtained as the limit

$$(f, g) = \lim_{h \rightarrow 0} \frac{fg \mp g f}{h}, \quad q = e^h = 1 + h + [h^2]. \quad (84)$$

It is straightforward to find

$$(z_a, z_b) = z_a z_b, \quad a < b, \quad (85)$$

$$(z_a, \bar{z}^b) = \begin{cases} z_a \bar{z}^b, & a \neq b \\ 2(1 + \sum_{c=1}^a z_c \bar{z}^c), & a = b \end{cases}, \quad (86)$$

$$(z_a, dz_b) = \begin{cases} z_a dz_b + 2z_b dz_a, & a < b \\ 2z_a dz_a, & a = b \\ z_a dz_b, & a > b \end{cases}, \quad (87)$$

$$(\bar{z}^a, dz_b) = \begin{cases} -\bar{z}^a dz_b, & a \neq b \\ -2 \sum_{c=1}^a \bar{z}^c dz_c, & a = b \end{cases} \quad (88)$$

and those following from the  $*$ -involution, which satisfies

$$(f, g)^* = (g^*, f^*). \quad (89)$$

The P.B. of two differential forms  $f$  and  $g$  of degrees  $m$  and  $n$  respectively satisfies

$$(f, g) = (-1)^{mn+1} (g, f). \quad (90)$$

The exterior derivatives  $\delta, \bar{\delta}, d$  act on the P.B.s distributively, for example

$$d(f, g) = (df, g) \pm (f, dg), \quad (91)$$

where the plus (minus) sign applies for even (odd)  $f$ . Notice that we have extended the concept of Poisson Bracket to include differential forms.

In the classical limit (61), (62) and (63) become

$$2\delta f = (\eta, f), \quad (92)$$

$$2\bar{\delta} f = (\bar{\eta}, f), \quad (93)$$

and

$$2df = (\Xi, f). \quad (94)$$

Equations (66), (67) and (71) to (83) are still valid, with  $q = 1$ , but now

$$\Xi^2 = 0. \quad (95)$$

The Fubini-Study Kähler form

$$K = dz_a g^{a\bar{b}} d\bar{z}^b \quad (96)$$

has vanishing Poisson bracket with all functions and forms and, naturally, it is closed. We find the validity of (92), (93) and (94) very remarkable, since the one-forms  $\eta$ ,  $\bar{\eta}$  and  $\Xi$  do not have to be adjoined to the space of one-forms but already belong there naturally.

## 5 Integration

We now turn to the discussion of integration on  $CP_q(N)$ . We shall use the notation  $\langle f(z, \bar{z}) \rangle$  for the right-invariant integral of a function  $f(z, \bar{z})$  over  $CP_q(N)$ . It is defined, up to a normalization factor, by requiring

$$\langle \mathcal{O}f(z, \bar{z}) \rangle = 0 \quad (97)$$

for any left-invariant vector field  $\mathcal{O}$  of  $SU_q(N+1)$ . In [1], the integral was computed for  $CP_q(1) = S_q^2$  by considering explicitly how the vector field act on functions. We shall follow a different and simpler approach here. First we notice that the identification

$$x_i/L^{1/2} = T_i^N, \quad \bar{x}^i/L^{1/2} = (T^{-1})_N^i, \quad i = 0, 1, \dots, N \quad (98)$$

reproduces (1)-(4). Thus if we define

$$\langle f(z, \bar{z}) \rangle \equiv \langle f(z, \bar{z}) |_{z_a = (T_0^N)^{-1} T_a^N, \bar{z}^a = (T^{-1})_N^a / (T^{-1})_N^0} \rangle_{SU_q(N+1)}, \quad (99)$$

where  $\langle \cdot \rangle_{SU_q(N+1)}$  is the Haar measure [10] on  $SU_q(N+1)$ , then it follows immediately that (97) is satisfied. ¶ Next we claim that

$$\langle (z_1)^{i_1} (\bar{z}^1)^{j_1} \dots (z_N)^{i_N} (\bar{z}^N)^{j_N} \rangle = 0 \text{ unless } i_1 = j_1, \dots, i_N = j_N. \quad (100)$$

This is because the integral is invariant under the finite transformation (57). For the particular choice  $T_j^i = \delta_j^i \alpha_i$ , with  $|\alpha_i| = 1$ ,  $\prod_{i=0}^N \alpha_i = 1$ , this gives

$$z_a \rightarrow (\alpha_a / \alpha_0) z_a \quad (101)$$

and so (100) follows.

In [10], Woronowicz proved the following interesting property for the Haar measure

$$\langle f(T)g(T) \rangle_{SU_q(N+1)} = \langle g(T)f(DTD) \rangle_{SU_q(N+1)} \quad (102)$$

where

$$(DTD)_j^i = D_k^i T_m^k D_j^m \quad (103)$$

and

$$D_j^i = q^{-N+2i} \delta_j^i \quad (104)$$

is the  $D$  matrix for  $SU_q(N+1)$ . It follows from (102) that

$$\langle f(z, \bar{z})g(z, \bar{z}) \rangle = \langle g(z, \bar{z})f(\mathcal{D}z, \mathcal{D}^{-1}\bar{z}) \rangle \quad (105)$$

where

$$\mathcal{D}_b^a = \delta_b^a q^{2a}, \quad a, b = 1, 2, \dots, N. \quad (106)$$

Introducing

$$\rho_r = 1 + \sum_{a=1}^r z_a \bar{z}^a, \quad (107)$$

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¶A similar strategy of using the “angular” measure to define an integration has been employed by H. Steinacker [9] in constructing integration over the Euclidean space.

one finds from (32) and (33) that

$$\rho_r z_a = \begin{cases} z_a \rho_r & r < a \\ q^{-2} z_a \rho_r & r \geq a \end{cases}, \quad (108)$$

$$\rho_r \rho_s = \rho_s \rho_r \quad (109)$$

and

$$\bar{z}^a z_a = q^{-2} \rho_a - \rho_{a-1} \quad (\text{no sum}). \quad (110)$$

Because of (100), it is sufficient to determine integrals of the form

$$\langle \rho_1^{-i_1} \cdots \rho_N^{-i_N} \rangle. \quad (111)$$

The values of the integers  $i_a$  for (111) to make sense will be determined later.

Consider

$$\begin{aligned} \langle \bar{z}_a \rho_1^{-i_1} \cdots \rho_N^{-i_N} z_a \rangle &= \langle \rho_1^{-i_1} \cdots \rho_N^{-i_N} z_a (q^{-2a} \bar{z}^a) \rangle \\ &= q^{-2a} \langle \rho_1^{-i_1} \cdots \rho_N^{-i_N} (\rho_a - \rho_{a-1}) \rangle \end{aligned} \quad (112)$$

Using (108),

$$\begin{aligned} \text{L.S.} &= q^{2(i_a + \cdots + i_N)} \langle \rho_1^{-i_1} \cdots \rho_N^{-i_N} \bar{z}^a z_a \rangle \\ &= q^{2I_a} \langle \rho_1^{-i_1} \cdots \rho_N^{-i_N} \bar{z}^a z_a \rangle \end{aligned} \quad (113)$$

where we have denoted

$$I_a = i_a + \cdots + i_N. \quad (114)$$

Using (110) we get the recursion formula

$$\begin{aligned} &\langle \rho_1^{-i_1} \cdots \rho_{a-1}^{-i_{a-1}+1} \rho_a^{-i_a} \cdots \rho_N^{-i_N} \rangle [I_a + a] \\ &= \langle \rho_1^{-i_1} \cdots \rho_{a-1}^{-i_{a-1}} \rho_a^{-i_a+1} \cdots \rho_N^{-i_N} \rangle [I_a + a - 1], \end{aligned} \quad (115)$$

where

$$[x] = \frac{q^{2x} - 1}{q^2 - 1}. \quad (116)$$

It is obvious then that

$$\langle \rho_1^{-i_1} \cdots \rho_a^{-i_a} \rangle = \langle \rho_1^{-i_1} \cdots \rho_{a-1}^{-i_{a-1}-i_a} \rangle \frac{[a]}{[I_a + a]}. \quad (117)$$

By repeated use of the recursion formula,  $\langle \rho_1^{-i_1} \cdots \rho_N^{-i_N} \rangle$  reduces finally to  $\langle \rho_1^{-i_1-i_2-\cdots-i_N} \rangle$  and

$$\langle \rho_1^{-I_1} \rangle = \frac{1}{[I_1 + 1]} \langle 1 \rangle. \quad (118)$$

Therefore

$$\langle \rho_1^{-i_1} \cdots \rho_N^{-i_N} \rangle = \langle 1 \rangle \prod_{a=1}^N \frac{[a]}{[I_a + a]}. \quad (119)$$

For this to be positive definite,  $i_a$  should be restricted such that  $I_a + a > 0$  for  $a = 1, \dots, N$ .

## 6 Braided $CP_q(N)$

The braiding of the  $C_q^{N+1}$  quantum planes induces a braiding on the  $CP_q(N)$ 's. Let the first copy of quantum plane be denoted by  $x_i, \bar{x}^i$  and the second by  $x'_i, \bar{x}'^i$ .

A consistent and covariant choice of commutation relations between them is

$$x_i x'_j = \tau \tilde{R}_{ij}^{kl} x'_k x_l, \quad (120)$$

$$\bar{x}^i x'_j = \nu (\tilde{R}^{-1})_{jl}^{ik} x'_k \bar{x}^l \quad (121)$$

and their  $*$ -involutions for arbitrary numbers  $\tau, \nu$ . If we choose  $\tau = \nu^{-1}$  then the Hermitian length  $L$  will remain central,  $L f' = f' L$ , for any function  $f'$  of  $x', \bar{x}'$ . However,  $L'$  does not commute with  $x, \bar{x}$ .

Assuming that the exterior derivatives of the two copies satisfy the Leibniz rule

$$\delta' f = \pm f \delta', \quad \bar{\delta}' f = \pm f \bar{\delta}', \quad (122)$$

$$\delta f' = \pm f' \delta, \quad \bar{\delta} f' = \pm f' \bar{\delta}, \quad (123)$$



where the plus (minus) signs apply for even (odd)  $f$  and  $f'$ , and

$$\delta\delta' = -\delta'\delta, \quad \delta\bar{\delta}' = -\bar{\delta}'\delta, \quad (124)$$

$$\bar{\delta}\delta' = -\delta'\bar{\delta}, \quad \bar{\delta}\bar{\delta}' = -\bar{\delta}'\bar{\delta}, \quad (125)$$

we obtain the commutation relations between functions and forms of different copies by letting  $\delta, \bar{\delta}, \delta'$  and  $\bar{\delta}'$  to act on (120) and (121). As usual, the commutation relations between derivatives and functions of different copies can also be derived from the commutation relations between differential forms and functions using the Leibniz rule of the exterior derivatives and the identifications  $\delta = dx_i D^i, \bar{\delta} = d\bar{x}^i \bar{D}_i$  for both copies.

From the above we derive the braiding relations of two braided copies of  $CP_q(N)$  in terms of the inhomogeneous coordinates. They are independent of the particular choice of  $\tau$  and  $\nu$ . We have

$$z_a z'_b = q \hat{R}_{ab}^{ce} (z'_c - q^{-1} \lambda z_c) z_e, \quad (126)$$

$$\bar{z}'^a z_b = q^{-1} (\hat{R}^{-1})_{be}^{ac} z_c \bar{z}'^e - q^{-1} \lambda \delta_b^a \quad (127)$$

and their  $*$ -involutions as well as the commutation relations between functions and forms of different copies following the assumption that their exterior derivatives anticommute.

## 7 Quantum Projective Geometry

We will show in the following that many concepts of projective geometry have an analogue in the deformed case, in particular we shall study the deformed anharmonic ratios (cross ratios).

### 7.1 Collinearity Condition

Classically the collinearity conditions for  $m$  distinct points can be given in terms of the inhomogeneous coordinates  $\{z_a^A | A = 1, 2, \dots, m; a = 1, 2, \dots, N\}$

as:

$$(z_a^A - z_a^B)(z_a^C - z_a^D)^{-1} = (z_b^A - z_b^B)(z_b^C - z_b^D)^{-1}, \quad (128)$$

where  $A \neq B, C \neq D = 1, \dots, m$  and  $a, b = 1, \dots, N$ .

In the deformed case, the coordinates  $\{z_a^A\}$  of  $m$  points must be braided for the commutation relations to be covariant, namely,

$$z_a^A z_b^B = q \hat{R}_{ab}^{ce} (z_c^B - q^{-1} \lambda z_c^A) z_e^A, \quad A \leq B, \quad (129)$$

as an extension of (126). Equation (127) can also be generalized in the same way, but we shall not need it in this section. This braiding has the interesting property that the algebra of  $CP_q(N)$  is *self-braided*, that is, (129) allows the choice  $A = B$ . This property makes it possible to talk about the coincidence of points. Actually, the whole differential calculus for braided  $CP_q(N)$  described in Sect.6 has this property.

Another interesting fact about this braiding is that for a fixed index  $a$  the commutation relation is identical to that for braided  $S_q^2$  <sup>||</sup>:

$$z_a^A z_a^B = q^2 z_a^B z_a^A - q \lambda z_a^A z_a^A, \quad A \leq B. \quad (130)$$

Since there is no algebraic way to say that two "points" are distinct in the deformed case, the collinearity conditions should avoid using expressions like  $(z_a^A - z_a^B)^{-1}$ , which are ill defined. Denoting

$$[AB]_a := z_a^A - z_a^B, \quad (131)$$

the collinearity conditions in the deformed case can be formulated as:

$$[AB]_a [CD]_b = q^2 [CD]_a [AB]_b, \quad \forall a, b, \quad (132)$$

and  $A < B \leq C < D$ . By (129) this equation is formally equivalent to the quantum counterpart of (128):

$$[AB]_a [CD]_a^{-1} = [AB]_b [CD]_b^{-1}, \quad (133)$$

---

<sup>||</sup>This formula differs from the corresponding one in [2] because in the present paper we have used different ordering conventions.

where the ordering of  $A, B, C, D$  is arbitrary. The advantage of this formulation is that (132) is a quadratic polynomial condition and polynomials are well defined in the braided algebra.

Therefore the algebra  $Q$  of functions of  $m$  collinear points is the quotient of the algebra  $A$  of  $m$  braided copies of  $CP_q(N)$  over the ideal  $I := \{f\alpha g | \forall f, g \in A; \forall \alpha \in CC\}$ , generated by  $\alpha$  which stands for the collinearity conditions (132), i.e.,  $\alpha \in CC := \{[AB]_a[CD]_b - q^2[CD]_a[AB]_b | A < B \leq C < D\}$ .

Two requirements have to be checked for this definition  $Q := A/I$  to make sense. The first one is that for any  $f \in A$  and  $\alpha \in CC$ ,

$$f\alpha = \sum_i \alpha_i f_i, \forall f \in A, \quad (134)$$

for some  $f_i \in A$  and  $\alpha_i \in CC$ . This condition ensures that the ideal  $I$  generated by the collinearity conditions is not "larger" than what we want, as compared with the classical case.

Note that not all the collinearity conditions are independent. In fact, it is sufficient (for formal manipulations, at least) to consider only  $B = C = m - 1, D = m$  in either (132) or (133). That is, we need only two points to fix a line.

We now check that (134) is satisfied. Obviously we only have to consider the cases  $f = z_c^E$ , for arbitrary  $E$  and  $c$ . Let  $\alpha(AB)_{ab} := [AB]_a[CD]_b - q^2[CD]_a[AB]_b$ , for  $C = m - 1$  and  $D = m$ . Using (129) one finds, after considerable algebra, for  $B \leq A < C < D$ ,

$$z_a^B \alpha(AC)_{bc} = q^2 \hat{R}_{ab}^{he} \hat{R}_{ec}^{fg} \alpha(AC)_{hf} z_g^B. \quad (135)$$

For  $A \leq B \leq C < D$ , one finds similarly

$$z_a^B \alpha(AC)_{bc} = q^2 \hat{R}_{ab}^{he} \hat{R}_{ec}^{fg} (\alpha(AC)_{hf} z_g^B + q^{-1} \lambda \alpha(AB)_{hf} [AB]_g). \quad (136)$$

Hence (134) is proven for  $B \leq C$ . Using

$$[CD]_a \alpha(AC)_{bc} = (\hat{R}^{-1})_{ab}^{he} (\hat{R}^{-1})_{ec}^{fg} \alpha(AC)_{hf} [CD]_g, \quad (137)$$

for  $B = D$  and

$$[BD]_a \alpha(AC)_{bc} = q^{-2} (\hat{R}^{-1})_{ab}^{he} (\hat{R}^{-1})_{ec}^{fg} \alpha(AC)_{hf} [BD]_g, \quad (138)$$

for  $B > D$ , together with the above two equations we immediately see that (134) is satisfied for  $f = z_a^B$  also for  $B \geq D$ . Therefore the first requirement is satisfied.

The second requirement is the invariance of  $I$  under the fractional transformations (57). While this can be directly checked for (132), it is equivalent but simpler to consider another expression of the collinearity conditions:

$$[AB]_a^{-1} [AB]_b = [CD]_a^{-1} [CD]_b, \quad (139)$$

where the ordering of  $A, B, C, D$  is arbitrary. Again we only have to consider the independent cases:  $B < A = C = m - 1, D = m$ . The fractional transformation has

$$[AB]_a \rightarrow -U(B)^{-1} [AB]_b z_c^A M_a^{bc} V(A)^{-1}, \quad (140)$$

where  $U(B) = T_0^0 + z_e^B T_0^e$ ,  $V(A) = T_0^0 + q z_f^A T_0^f$  and  $M_a^{bc} = T_0^b T_a^c - q T_a^b T_0^c$ . So

$$\begin{aligned} & [AB]_a^{-1} [AB]_b \rightarrow \\ & V(A) ([AB]_c z_h^A M_a^{ch})^{-1} ([AB]_e z_f^A M_b^{ef}) V(A)^{-1} \\ = & V(A) ([AB]_c [AC]_c^{-1} [AC]_c z_h^A M_a^{ch})^{-1} ([AB]_e [AC]_e^{-1} [AC]_e z_f^A M_b^{ef}) V(A)^{-1} \\ = & V(A) ([AC]_c z_h^A M_a^{ch})^{-1} ([AB]_g [AC]_g^{-1})^{-1} ([AB]_s [AC]_s^{-1}) ([AC]_e z_f^A M_b^{ef}) V(A)^{-1} \\ = & V(A) ([AC]_c z_h^A M_a^{ch})^{-1} ([AC]_e z_f^A M_b^{ef}) V(A)^{-1}, \end{aligned} \quad (141)$$

(where we used (133) for the second equality) which equals the transformation of  $[AC]_a^{-1} [AC]_b$ . This means that the relation  $[AB]_a^{-1} [AB]_b - [AC]_a^{-1} [AC]_b = 0$  is preserved by the transformation.

## 7.2 Anharmonic Ratios

Classically the anharmonic ratio of four collinear points is an invariant of the projective mappings, which are the linear transformations of the homogeneous coordinates. In the deformed case, the homogeneous coordinates are

the coordinates  $x_i, \bar{x}_i$  of the  $SU_q(N+1)$ -covariant quantum space, and the linear transformations are the  $GL_q(N+1)$  transformations \*\* (26), which induce the fractional transformations (57) on the coordinates  $z_i, \bar{z}_i$  of the projective space  $CP_q(N)$ .

We define the anharmonic ratio of  $CP_q(N)$  for four collinear points  $\{z_a^A | A = 1, 2, 3, 4\}$  to be

$$[A1]_a[A4]_a^{-1}[B4]_a[B1]_a^{-1}, \quad (142)$$

where  $A, B = 2, 3$ . We wish to show that it is invariant. Using (140) and denoting  $\tau(A) := [1A]_a[14]_a^{-1}$  which is independent of the index  $a$  according to the collinearity condition, we get

$$[AB]_a \rightarrow U(B)^{-1}(\tau(A) - \tau(B))P_a(A)V(A)^{-1}, \quad (143)$$

where  $P_a(A) := [14]_b z_c^A M_a^{bc}$ . Then the anharmonic ratio (142) transforms as

$$\begin{aligned} [A1]_a[A4]_a^{-1}[B4]_a[B1]_a^{-1} &\rightarrow U(1)^{-1}\tau(A)(1 - \tau(A))^{-1}(1 - \tau(B))\tau(B)^{-1}U(1) \\ &= \tau(A)(1 - \tau(A))^{-1}(1 - \tau(B))\tau(B)^{-1}, \\ &= [A1]_a[A4]_a^{-1}[B4]_a[B1]_a^{-1}, \end{aligned} \quad (144)$$

where we have used  $z_a^1 \tau(A) = \tau(A) z_a^1$  for any  $A$ , which is true because we can represent  $\tau(A)$  as  $[1A]_a[14]_a^{-1}$  with the same index  $a$  and then use  $z_a^1[AB]_a = q^2[AB]_a z_a^1$ .

Because of the nice property (130), we can use the results about the anharmonic ratios of  $S_q^2$  ( which is a special case of  $CP_q(N)$  with  $N = 1$  but no collinear condition is needed there) in [1]. Note that all the invariants as functions of  $z_a^A$  for a fixed  $a$  in  $CP_q(N)$  are also invariants as functions of  $z^A = z_a^A$  in  $S_q^2$ . The reason is the following. Consider the matrix  $T_b^a$  defined by

$$T_0^0 = \alpha, \quad T_a^0 = \beta, \quad (145)$$

$$T_0^a = \gamma, \quad T_a^a = \delta, \quad (146)$$

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\*\*The commutation relations (1) are also covariant under  $GL_q(N+1)$  transformations.

where  $\alpha, \beta, \gamma, \delta$  are components of an  $SU_q(2)$  matrix, and  $T_b^b = 1$  for all  $b \neq a$ , both  $> 0$ , with all other components vanishing. It is a  $GL_q(N+1)$  matrix, but the transformation (57) of  $z_a^A$  by this matrix is the fractional transformation on  $S_q^2$  with coordinate  $z^A = z_a^A$ .

Therefore all the anharmonic ratios of  $CP_q(N)$  must have a corresponding anharmonic ratio of  $S_q^2$ . On the other hand, since all the anharmonic ratios of  $S_q^2$  are functions of only one of them [2], all them have a corresponding invariant of  $CP_q(N)$ , which are functions of (142) and may also be called anharmonic ratios. Hence we have established a one to one correspondence between the anharmonic ratios of  $S_q^2$  and  $CP_q(N)$ , and as a consequence the fact that all the anharmonic ratios of  $CP_q(N)$  are functions of only one of them.

The anharmonic ratios are important because they are the building blocks of invariants in projective geometry. For example, in the  $n$ -dimensional classical case for given  $2(n+1)$  points with homogeneous coordinates  $\{x_i^A\}$ , inhomogeneous coordinates  $\{z_a^A\}$  where  $A = 1, \dots, n$ ,  $i = 0, 1, \dots, n$ , and  $a = 1, \dots, n$ , we can construct an invariant

$$I := \frac{(1, 2, \dots, n, n+1)(n+2, n+3, \dots, 2(n+1))}{(1, 2, \dots, n, n+2)(n+1, n+3, \dots, 2(n+1))} \quad (147)$$

where  $(A_0, \dots, A_n)$  is the determinant of the matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_1^{A_0} & \cdots & z_1^{A_n} \\ \vdots & \ddots & \vdots \\ z_n^{A_0} & \cdots & z_n^{A_n} \end{pmatrix}. \quad (148)$$

It is invariant because  $(A_0, \dots, A_n)$  equals the determinant of the matrix  $M_j^i = x_j^{A_i}$ ,  $i, j = 0, \dots, n$ , divided by the factor  $x_0^{A_0} \cdots x_0^{A_n}$ , which cancels between the numerator and denominator of  $I$ . It can be shown that this invariant  $I$  is in fact the anharmonic ratio of four points  $z, z', z^{n+1}, z^{n+2}$ , where  $z$  ( $z'$ ) is the intersection of the line fixed by  $z^{n+1}, z^{n+2}$  with the  $(n-1)$ -dimensional subspace fixed by  $z^1, \dots, z^n$  ( $z^{n+3}, \dots, z^{2(n+1)}$ ).

It is remarkable that all this can also be done in the quantum case. We can construct an invariant  $I_q$  using the quantum determinant and we can also formulate the condition for  $(n + 1)$  points to share an  $(n - 1)$ -dimensional subspace. Furthermore, we know how to describe the intersection between subspaces of arbitrary dimension spanned by given points. It can be shown that the invariant  $I_q$  is indeed an anharmonic ratio in the same way as the classical case.

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